Midterm 1

EC031-S25

Name: ______________________________

For this midterm, you will be asked several questions about what we've learned in class. There are five questions in this exam. **Choose and answer 4 out of 5 questions.** You will not receive extra credit for answering all questions.

The exam is out of 100 points. By default, all math-involved questions require showing your work, unless otherwise stated. Simply writing down the right answer is not enough for full credit.

For "explain your answer" questions, the length of the answer is not as important as being able to clearly explain your thinking. As a general rule, 3-5 sentences should suffice.

The amount of space that I give you is not indicative of how much is needed to answer the question. Clearly mark the questions that you choose.

Tip: Before starting the exam, look through all the questions and choose the ones you want to answer.

Problem 1 [25 points]

A consulting firm submitted a bid for a large research project. The firm's management initially felt they had a 50–50 chance of getting the project. However, the agency to which the bid was submitted subsequently requested additional information on the bid. Past experience indicates that for 75% of the successful bids and 40% of the unsuccessful bids the agency requested additional information.

a. [5 points] What is the prior probability of the bid being successful (that is, prior to the request for additional information)?

$P(\text{success}) = .5$

b. [~~10~~ 5 points] What is the conditional probability of a request for additional information given that the bid will ultimately be successful?

$$P(\text{Information} \mid \text{Success}) = .75$$

c. [~~10~~ 15 points] Compute the posterior probability that the bid will be successful given a request for additional information.

$$P(\text{Success} \mid \text{Information}) =$$

$$\frac{P(\text{Information} \mid \text{Success})\,P(\text{Success})}{P(\text{Information})}$$

$$= \frac{P(\text{Information} / \text{Success})\,P(\text{Success})}{P(\text{Information} / \text{Success})\,P(\text{Success}) + P(\text{Information} \mid \text{Fail})\,P(\text{Fail}}$$

$$= \frac{.75(.5)}{.75(.5) + .4(.5)} = .65$$

Problem 2 [25 points]

You are deciding whether to play a proposed game with your friend. Here are the rules:

You roll a die and if it lands on a 3, then I will pay you \$30.

If it doesn't land on a 3, then you pay me \$5.

a. [10 points] What are your *expected* winnings (i.e. what is the expected value of playing the game) once?

$$E(x) = \frac{1}{6}(30) + \frac{5}{6}(-5)$$

$$= \frac{30}{6} - \frac{25}{6} = \boxed{\frac{5}{6}}$$

b. [5 points] Let's say you play this game 5 times. What distribution would you use to find the probability of winning x times? write down this distribution for this situation.

<u>Binomial:</u>

$$f(x) = \binom{5}{x}\left(\frac{1}{6}\right)^{x}\left(\frac{5}{6}\right)^{5-x}$$

c. [10 points] Using the distribution you chose in (b), what is the probability of winning at least once?

$$P(x \geq 1) = 1 - P(x=0)$$

$$P(x=0) = \binom{5}{0}\left(\frac{1}{6}\right)^0\left(\frac{5}{6}\right)^5$$

$$= \frac{5!}{0!\,5!}(1)\cdot\left(\frac{5}{6}\right)^5 = .40$$

$$P(x \geq 1) \approx 1 - .40 \approx .6$$

d. [5 points] Would you play this game? Explain your answer.

Since each game is independent & each has positive winnings, you might as well play!

Problem 3 [27 points]

Suppose you want to want to find out how what the wage in the labor market will be next year. You know that the wage is normally distributed with a mean of \$50,000 and a standard deviation of \$10,000. What is the probability that the wage will be:

(Hint: Remember that the normal distribution is symmetric.)

a. [5 points] Less than \$40,000?

$$P(x \leq 40{,}000)$$

$$= P\left(z \leq \frac{40{,}000 - 50{,}000}{10{,}000}\right)$$

$$= P(z \leq -1) = \boxed{0.1587}$$

b. [6 points] More than \$60,000?

$$P(x > 60{,}000)$$

$$= P\left(z > \frac{60{,}000 - 50{,}000}{10{,}000}\right)$$

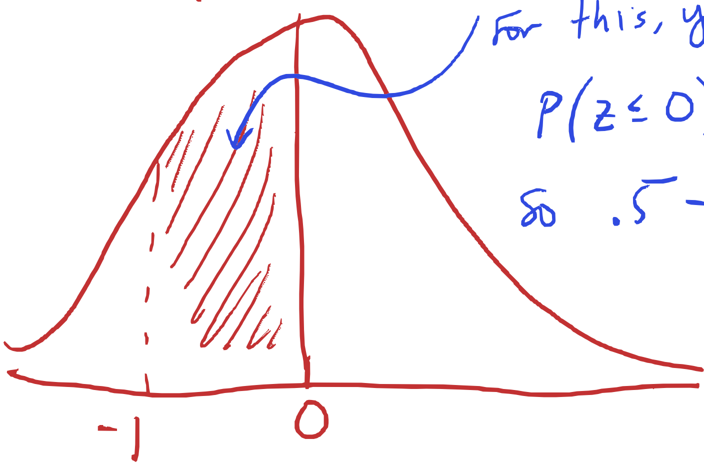
$$= P(z > 1)$$

By symmetry: $P(z \leq -1) = P(z \geq 1) = P(z > 1) + P(z = 1)$
$= P(z > 1)$

$$= \boxed{0.1587}$$

c. [6 points] Between \$40,000 and \$50,000?

$$P(40{,}000 \le x \le 50{,}000) = P(-1 \le z \le 0)$$



For this, you can do:

$$P(z \le 0) - P(z \le -1)$$

so $.5 - .1587 \approx \boxed{.34}$

d. [10 points] What is the 90th percentile of the wage distribution?

We need to find a wage, w, such that

$$P\left(z \le \frac{w - 50000}{10{,}000}\right) = .9$$

From the sheet we know that

$$\frac{w - 50{,}000}{10{,}000} = 1.28$$

$$\Rightarrow \boxed{w = 50{,}000 + 10{,}000(1.28)}$$

Problem 4 [25 points]

You would like to better understand the time between arrivals of the ambulance at hospital emergency rooms. In the past you've calculated that it takes, on average, 8 minutes between arrivals. You decide to model this as an exponential distribution.

a. [5 points] What is the probability that the ambulance will arrive in less than 6 minutes?

$$P(x \leq 6) = 1 - e^{-6/8} = 1 - .4724 = \boxed{.5276}$$

b. [5 points] What is the probability that the ambulance will arrive in more than 6 minutes?

$$1 - P(x \leq 6) = 1 - .5276 = \boxed{.4724}$$

c. [5 points] What is the probability that the ambulance will arrive in between 4 and 6 minutes?

$$P(4 \leq x \leq 6) = P(x \leq 6) - P(x \leq 4)$$

$$= .5276 - .3935 = \boxed{.1341}$$

d. [10 points] One of your colleagues says:

Why are you using the exponential distribution to model the time between emergency vehicle arrivals? Don't you know that things are normally distributed in the end? It's the central limit theorem...

How would you answer your colleague?

Your colleague is misapplying the CLT. The CLT applies to the distribution around the mean, not the random variable itself.

Problem 5 [30 points]

Suppose you want to understand how the onset of rain changes the number of car accidents. Suppose the following table shows the joint distribution between whether it rained in the last six months and the probability of an accident happening.

	Rain (r=1)	No Rain (r=0)
No accident (a=0)	0.2	0.1
Accident (a=1)	0.4	0.3

a. [1 points] What is the probability of an accident occurring?

.7

b. [1 points] What is the probability of no accident occurring?

.3

c. [1 points] What is the probability of rain?

.6

d. [1 points] What is the probability of it not raining?

$.4$

e. [1 points] What is the probability of an accident occurring and it raining?

$.4$

f. [5 points] What is the probability of an accident occurring, given that it rained?

$$P(\text{Accident} \mid \text{rain}) = \frac{P(\text{Accident} \cap \text{rain})}{P(\text{rain})}$$

$$= \frac{.4}{.6} = \frac{4}{6} = \frac{2}{3} = .667$$

g. [5 points] What is the probability of an accident occurring, given that it did not rain?

$$P(\text{Accident} \mid \text{no rain})$$

$$= \frac{P(\text{Accident} \cap \text{no rain})}{P(\text{no rain})}$$

$$= \frac{.3}{.4} = \frac{3}{4} = .75$$

h. [5 points] Are the events of an accident occurring and it raining independent? Explain your answer and show your work.

Are these two random variables independent?

We can check by checking if:

$$p(x, y) = p(x)p(y)$$

$p(\text{accident}, \text{rain}) = .4$

$p(\text{accident})p(\text{rain}) = .7(.6) = .42 \neq .4$

$\therefore$ Not independent

You could so check if

$P(\text{accident} | \text{rain}) = P(\text{accident})$

$.667 \neq .7$

i. [10 points] Calculate the covariance between the two variables.

For this we first need μ_r & μ_a:

$$\mu_r = p(r=0)\cdot(0) + p(r=1)\cdot 1 = .6$$

$$\mu_a = p(a=0)\cdot 0 + p(a=1)\cdot 1 = .7$$

$$Cov(r,a) = \sum_r \sum_a (r-\mu_r)(a-\mu_a)\, p(r,a)$$

$$= (0-.6)(0-.7)(.2)$$

$$+ (0-.6)(1-.7)(.4)$$

$$+ (1-.6)(0-.7)(.1)$$

$$+ (1-.6)(1-.7)(.3)$$

$$= 0 + (-.4)(.3)(.4) + (.4)(-.3)(.1)$$

$$+ (.4)(.3)(.3)$$

$$= (.4)(.3)\left[-.4 - .1 + .3\right] = .12\cdot(-.2)$$

$$\boxed{= -.02}$$

Equation Sheet

Binomial Distribution

$$f(x) = \frac{n!}{x!(n-x)!} \cdot p^x (1-p)^{(n-x)}$$

where n is the number of trials, x is the number of successes, and p is the probability of success.

Exponential Distribution

$$f(x) = \frac{1}{\mu} e^{-\frac{1}{\mu} x}$$

where μ is the mean of the distribution.

Poisson Distribution

$$f(x) = \frac{\mu^x e^{-\mu}}{x!}$$

where μ is the mean of the distribution.

Combinations

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Permutations

$$P(n,k) = \frac{n!}{(n-k)!}$$

The Weighted Mean

$$\bar{x} = \frac{\sum_{i=1}^{n} w_i x_i}{\sum_{i=1}^{n} w_i}$$

where w_i is the weight of the ith observation.

Normal Distribution Values

For the Normal Distribution, the following probabilities are given:

$$P(x \leq -1) = 0.1587, \quad P(x \leq -2) = 0.0228, \quad P(x \leq -3) = 0.0013$$

$$P(x \leq 1.28) = .9$$

Expected Value

$$E(x) = \sum_{i=1}^{n} x_i p_i$$

where x_i is the value of the random variable and p_i is the probability of that value.

Variance

$$Var(x) = \sum_{i=1}^{n} (x_i - \mu)^2 p_i$$

where x_i is the value of the random variable, μ is the mean of the distribution, and p_i is the probability of that value.

Covariance

$$Cov(x, y) = \frac{\sum_{i=1}^{n} (x_i - \mu_x)(y_i - \mu_y)}{n}$$

where x_i and y_i are the values of the random variables, μ_x and μ_y are the means of the distributions, and n is the number of observations.